Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization

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1 Introduction

In 1847, British mathematician George Boole (1815–1864) published a work entitled *The Mathematical Analysis of Logic*; seven years later he further developed his mathematical approach to logic in *An Investigation of the Laws of Thought*. Boole's goal in these works was to extend the boundaries of traditional logic by developing a general method for representing and manipulating logically valid inferences[1]. To this end, Boole employed letters to represent classes (or sets) and defined a system of symbols $(\times, +)$ to represent operations on these classes. As illustrated in the following excerpt from *Laws of Thought* [3, pp. 24–38], his definition of logical multiplication xy on classes corresponded to today's operation of set intersection:

> Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep."

Boole's definition of logical addition $x+y$ as the aggregate of class x with class y in turn corresponded, in essence, to today's operation of set union.[2]

As suggested by the title of his later work, Boole's primary interest was in the fundamental laws which these operations on classes obey. Many of these laws, he discovered, were analogous to laws for standard algebra. For example, logical multiplication is commutative $[xy = yx]$ since [i]t is indifferent in what order two successive acts of election are performed.' That is, whether we select the white things from the class of sheep $[xy]$ or we select the sheep from the class of white things $[yx]$, we obtain the same result: the class of white sheep. In a similar fashion, one can use Boole's definitions of $+$ and $\times$ for classes to show that logical multiplication is associative $[z(xy) = (zx)y]$, that logical addition is commutative $[x + y = y + x]$ and associative $[x + (y + z) = (x + y) + z]$, and that logical multiplication is distributive over logical addition $[x(y + z) = xy + xz]$ — just as occurs in standard algebra.

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[1] For further details on Boole's work in logic and modifications made to it by John Venn (1834–1923) and C. S. Peirce (1839–1914), see the project "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce," Janet Barnett author.

[2] For various technical reasons, Boole restricted his use of + to classes which were disjoint. Most of his immediate followers, however, relaxed this restriction, so that their use of + corresponded exactly to today's operation of set union. British mathematician John Venn (1834–1923) discussed this issue in detail in the second (1894) edition of his *Symbolic Logic* [**?**, pp. 42-46]. Ultimately, Venn adopted an unrestricted use of + 'partly . . . because the voting has gone this way, and in a matter of procedure there are reasons for not standing out against such a verdict . . . '.

Other laws of Boole's 'Algebra of Logic' differed substantially from those of standard algebra, however. Among these is the *Idempotent Law*: $x^2 = x$. As noted by Boole, this equation holds in standard algebra only when $x = 0$ or $x = 1$. He further commented [3, p. 47] that for

> . . . an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone . . . the laws, the axioms, and the processes . . . will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic.

In the mid-twentieth century, this special two-valued 'arithmetical algebra' became important in the application of boolean algebra to the design of circuits[3]; later in this project, we will also see how this same two-valued algebra was used in the early twentieth century work on axiomatization of boolean algebras. Boole, however, primarily used the symbols '0' and '1' to denote two special classes — 'nothing' ('empty set') and 'universe' ('universal set') respectively — in the Algebra of Logic.[4] But within the Algebra of Logic itself, Boole proved that the Idempotent Law ($x^2 = x$) is a completely general law which holds for *all* classes x, *not* just for these special two.[5] In addition to establishing this and other unusual new laws, Boole showed that the Algebra of Logic fails to satisfy certain well-known laws of standard algebra. For example, the equation $zx = zy$ does *not* imply that $x = y$, even in the case where $z \neq 0$![6]

In short, although explicitly and intentionally algebraic in nature, Boole's method of logical analysis led to a very strange new system of algebra. This project examines the subsequent development of this algebraic system, known today as a *boolean algebra*, in the hands of the generation of mathematicians which followed that of which Boole was a member. As a representative of this later generation and their style, we focus on the axiomatic framework proposed by Edward V. Huntington (1874-1952) in his 1904 paper *Sets of Independent Postulates for the Algebra of Logic*.

2 Axiomatization of Abstract Boolean Algebras

American-born Huntington (1874-1952) was educated at Harvard where he completed both a bachelor's and master's degree in mathematics. At the time, it was not unusual for American mathematicians to complete their doctoral studies in Europe. Huntington completed his Ph.D. in Strasbourg, then a part of Germany, in 1901; his thesis work was in algebra. Upon his return to the United States, he began a 40-year academic career at Harvard; his 1909 marriage produced no children. During the last half of his tenure at Harvard, Huntington served as professor of mechanics, a position in which he was able to explore his interest in mathematical teaching methods for engineers. He was also active in founding the Mathematical Association of America, and held offices in both the American Mathematical Society and

[3]For further details, see the project "Applying Boolean Algebra to Circuit Design: Claude Shannon," Janet Barnett author.

[4]To justify the use of the symbols '0' and '1' to represent the classes 'nothing' and 'universe,' Boole used the analogy between the roles played by these numbers in algebra and the roles played by these special classes in logic [3, p. 47–48]. He argued, for example, that since $0y = 0$ in standard algebra, then '. . . we must assign to the symbol 0 such an interpretation that the class represented by $0y$ may be identical with the class represented by 0, whatever the class y may be. A little consideration will show that this condition is satisfied if the symbol 0 represent Nothing.' A similar analysis of the algebraic equation $1y = y$ led him to conclude that '. . . the class represented by 1 must be "the Universe," since this is the only class in which are found all the individuals that exist in any class.'

[5]For Boole, the validity of the Idempotent Law for all classes followed directly from the definition of xy as 'the whole of that class of objects to which the names or qualities represented by x and y are together applicable ', from which fact 'it follows that if the two symbols have exactly the same signification, their combination expresses no more than either of the symbols taken alone would do.' (See [3, p. 31].) Selecting the sheep from the class of sheep, for instance, gives us just the class of sheep, so that $x\,x = x$, or $x^2 = x$.

[6]Exercise for reader: Provide a specific example of classes x , y and z to show that it is possible to have $zx = zy$ with $x \neq y$ and $z \neq 0$.

the American Association for the Advancement of Science. Although Huntington's scholarly legacy is less well known now than it was during his own life time, he is remembered today for his contribution to the study of mathematical methods of apportionment; the method of appointing representatives to the US Congress used today is due to him. Huntington's primary research interest, however, lay in what was then a fairly new field: the axiomatic foundations of algebraic systems.

The formal style of presentation and level of rigor displayed in Huntington's writing is indicative of a general shift in mathematics that began in the nineteenth century. In the latter half of that century, mathematicians witnessed a number of amazing new discoveries — geometries in which Euclid's Parallel Postulate was violated, algebras in which basic laws like commutativity failed or strange new laws such as Boole's 'idempotent law' $x^2 = x$ held, 'transfinite' number systems involving different sizes of infinity. These discoveries in turn encouraged mathematicians to introduce even greater levels of abstraction in their work. Huntington's 1904 paper fully encapsulated the formal rigor which became a hallmark of twentieth-century mathematics as mathematicians endeavored to make sense of the strange new wonders of their world. In fact, he opened the paper with a proclamation of his intent to disregard applications in favor of a completely formal development [5, p. 288]:

> The algebra of symbolic logic, as developed by LEIBNIZ, BOOLE, C.S. PEIRCE, E. SCHRÖDER, and others is described by WHITEHEAD as "the only known member of the non-numerical genus of universal algebra." This algebra, although originally studied merely as a means of handling certain problems in the logic of classes and the logic of propositions, has recently assumed some importance as an independent calculus; it may therefore be not without interest to consider it from a purely mathematical or abstract point of view, and to show how the whole algebra, in its abstract form, may be developed from a selected set of fundamental propositions, or postulates, which shall be independent of each other, and from which all the other propositions of the algebra can be deduced by purely formal processes.
>
> In other words, we are to consider the construction of a purely *deductive theory*, without regard to its possible applications.

Before proposing specific sets of postulates for the algebra of symbolic logic (i.e., boolean algebra), Huntington provided a more general description of the key components of a 'deductive system.' As you read his discussion of 'fundamental concepts' below [5, p. 288-290], be sure to also read his footnotes (7, 8 and 9) which further describe the concepts in question.

> The first step in such a discussion is to decide on the *fundamental concepts* or *undefined symbols*, concerning which the statements of the algebra are to be made. ... ...
>
> One such [fundamental] concept, common to every mathematical theory, is the notion of
>
> 1) a *class (K) of elements* $(a, b, c, \cdots)$.[7] ... ...

[7] Huntington's footnote: A *class* is determined by stating some condition which every entity in the universe must either satisfy or not satisfy; every entity which satisfies the condition is said to *belong* to the class. (If the condition is such that no entity can satisfy it, the class is called a "null" class.) Every entity which belongs to the class in question is called an *element* (cf. H.WEBER, *Algebra*, vol. 2 (1899), p.3).

In regard to the other fundamental concepts, one has usually a considerable freedom of choice; . . . . . . for the algebra of logic the fundamental concepts (besides the notion of *class*) may be selected at pleasure from the following:

2) a *rule of combination*[8], denoted, say, by $\oplus$ (read, for convenience, "plus"; see remark on these symbols below);

3) another *rule of combination*, denoted, say, by $\odot$ (read, "times");

4) a *dyadic relation*[9], denoted, say, by $\otimes$.

Notice that Huntington's operations do not apply to classes, as was the case with Boole's operations; rather, his operations apply to *undefined* elements (a , b , c , $\cdots$) from a single given class K. Neither the elements nor the operations possess any particular meaning — they are merely symbols. This marks a significant departure from earlier work on symbolic logic in which a concrete interpretation (e.g., + to represent aggregation of classes) was used not merely as a guide in developing appropriate laws, but to actually prove that these laws held in the algebra. Early in his paper, Huntington does mention that the 'class of regions in the plane' (i.e., sets as represented by a Venn diagram) is one possible concrete interpretation of his intended algebra [5, p. 292]. However, he did so only after emphasizing that 'the algebra is necessarily treated here solely in its abstract form, without reference to its possible application — that is, without reference to the possible interpretations of the symbols K, $\oplus$, $\odot$ and $\otimes$' [5, p. 290].

In short, Huntington did not set out to deduce the algebraic laws of a particular interpretation. Rather, his goal was to *deduce the algebra* from a set of laws which he assumed as postulates. It is these postulates that give meaning to the operations and elements of the algebra, as Huntington described in the following excerpt [5, p. 291]:

Having chosen the fundamental concepts, the next step is to decide on the *fundamental propositions* or *postulates*, which are to stand at the basis of the algebra. These postulates are simply *conditions* arbitrarily imposed on the fundamental concepts and must not, of course, be inconsistent among themselves. Any set of *consistent* postulates would give rise to a corresponding algebra, — namely the totality of propositions which follow from these postulates by logical deduction.

. . . .

Remarks on the symbols $\oplus$, $\odot$, *etc.* The symbols $\oplus$, $\odot$, and $\otimes$ are chosen with a double object in view. On account of the circles around them they are sufficiently unfamiliar to remind us of their true character as undefined symbols which have no properties not expressly stated in the postulates; while the $+, \cdot,$ and $<$ within the circles enable us to adopt, with the least mental effort, the interpretation which is likely to be the most useful. . . .

[8] Huntington's footnote: A *rule of combination* $\circ$, in the given class, is a convention according to which every two elements a and b (whether $a = b$ or $a \neq b$), in a definite order determine uniquely an entity $a \circ b$ (read "a with b"), which is, however, not necessarily an element of the class. In the class of quantities or numbers, familiar examples of rules of combination are $+, -, \times, \div$, etc.

[9] Huntington's footnote: A *dyadic relation*, R, in the given class, is determined when, if any two elements a and b are given in a definite order, we can decide whether a stands in the relation R to b or not; if it does, we write $a\mathrm{R}b$ . . . In the class of quantities or numbers, familiar examples of dyadic relations are $=, <, >, \leq$, etc. Relations among human beings furnish other examples.

The symbols $\bigwedge$ and $\bigvee$, which occur below, I take from PEANO's Formulaire de Mathématiques, vol. 4 (1903), pp. 27-28. The resemblance which these symbols bear to an empty glass and a full glass will facilitate the interpretation of the as "nothing" and "everything" respectively.

1. As suggested by his remarks on the symbols '$\oplus$' and '$\odot$', Huntington deliberately selected these as operation symbols to remind his readers of the class operations first studied by Boole[10], with '$+$' for logical addition (set union) and '$\times$' for logical multiplication (set intersection).

   (a) Based on Huntington's remarks on the symbols '$\bigwedge$' and '$\bigvee$', how do these two symbols relate to the special classes which Boole denoted by the symbols '0' and '1'?

   (b) In light of the correspondence between $(\oplus, \odot, \bigwedge, \bigvee)$ and $(+, \times, 0, 1)$ suggested by Huntington's remarks, identify 2-4 algebraic properties which you think $\bigwedge$ and $\bigvee$ should satisfy in Huntington's system. Write each property in both Huntington's notation $(\oplus, \odot, \bigwedge, \bigvee)$ and in Boole-like notation $(+, \times, 0, 1)$, and explain (briefly) why you think each of these properties should hold.

In fact, Huntington gave three separate such sets of postulates for his algebra, for reasons that we will consider briefly below. We consider only the first of these in detail [5, p. 292-293]:

§1. The First Set of Postulates.

Ia. *$a \oplus b$ is in the class whenever a and b are in the class.*

Ib. *$a \odot b$ is in the class whenever a and b are in the class.*

IIa. *There is an element $\bigwedge$ such that $a \oplus \bigwedge = a$ for every element a.*

IIb. *There is an element $\bigvee$ such that $a \odot \bigvee = a$ for every element a.*

IIIa. *$a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ are in the class.*

IIIb. *$a \odot b = b \odot a$ whenever a, b, $a \odot b$, and $b \odot a$ are in the class.*

IVa. *$a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ whenever a, b, $a \oplus b$, $a \oplus c$, $b \odot c$, $a \oplus (b \odot c)$ and $(a \oplus b) \odot (a \oplus c)$ are in the class.*

IVb. *$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ whenever a, b, $a \odot b$, $a \odot c$, $b \oplus c$, $a \odot (b \oplus c)$ and $(a \odot b) \oplus (a \odot c)$ are in the class.*

V. If the elements $\bigwedge$ and $\bigvee$ in postulates IIa and IIb exist and are unique, then for every element a there is an element $\overline{a}$ such that $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$

VI. There are at least two elements, x and y, in the class such that $x \neq y$.

[10]Huntington's dyadic relation symbol $\oslash$ is intended to suggest the relation of class (or set) inclusion. Boole himself did not consider a dyadic relation in his work on logic. A systematic study of the relation 'inclusion' for classes and the corresponding relation 'material implication' in propositional logic was first conducted by the American mathematician and philosopher Charles Sanders Peirce (1839–1914). In today's notation, 'inclusion' is denoted by the symbol '$\subseteq$' [which is read 'is contained in'] and 'material implication' is denoted by the symbol '$\rightarrow$' [which is read ' If ..., then ...']. Peirce denoted both relations by the single symbol '$\prec$'. For further details on the topic of material implication in propositional logic, see the project "Deduction through the Ages: A History of Truth," Jerry Lodder author.

2. Note that Postulates Ia, IIa and IIIa correspond to the following algebraic properties:

Ia.	Closure of K under Addition ($\oplus$)
IIa.	Identity ($\bigwedge$) for Addition ($\oplus$)
IIIa.	Commutativity of Addition ($\oplus$)

Identify the names of the algebraic properties given by Postulates Ib, IIb, IIIb, IVa and IVb.

3. Using today's symbolic notation[11], we could write Postulates Ia, IIa and IIIa as follows:

Ia. $(\forall a, b)(a \in K \wedge b \in K \to a \oplus b \in K)$

IIa. $(\exists \bigwedge \in K)(\forall a \in K)(a \oplus \bigwedge = a)$

IIIa. $(\forall a, b)\,[(a \in K) \wedge (b \in K) \wedge (a \oplus b \in K) \wedge (b \oplus a \in K) \to a \oplus b = b \oplus a]$

(a) More succinctly, we could also write Postulate Ia as follows: $(\forall a, b \in K)(a \oplus b \in K)$ Explain why it would *not* be legitimate to abbreviate Postulate IIIa as follows:

$$(\forall a, b \in K)(a \oplus b = b \oplus a)$$

What additional assumption could we make so that this would be a legitimate symbolic abbreviation of Postulate IIIa? Explain. (Hint? Review footnote 8, page 4 of this project.)

(b) Would it be legitimate to reverse the order of quantifiers in Postulate IIa as follows?

$$(\forall a \in K)(\exists \bigwedge \in K)(a \oplus \bigwedge = a)$$

Explain clearly why or why not.

(c) Write Postulates Ib, IIb, IIIb and VI in symbolic notation (see footnote 11 below).

4. This question examines the general concept of uniqueness which arises in Huntington's Postulate V (re-stated below for convenience).

V. If the elements $\bigwedge$ and $\bigvee$ in postulates IIa and IIb exist and are unique, then for every element a there is an element $\overline{a}$ such that $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$.

Also recall that Postulates IIa, b assert $\bigwedge$ is an additive identity and $\bigvee$ a multiplicative identity.

(a) The symbol '$\overline{a}$' was also used by some earlier mathematicians to denote the class not-a in the logic of classes.[12] Suppose we wish to interpret Huntington's postulates as statements about operations on classes. Use a specific example of a class a to illustrate the meaning of Postulate V within this interpretation. Indicate clearly what the classes a, not-a, 0 and 1 represent in your example. (It may be useful to review project question 1, page 5.)

[11] '$\forall$' represents the universal quantifier 'for all' ; '$\exists$' represents the existential quantifier 'there exists' ; '$\to$' represents the conditional 'if ..., then ...' ; '$\wedge$' represents the conjunction 'and' ; '$\vee$' represents the disjunction 'or'

[12] Boole himself denoted the class not-a by the expression '$1 - a$' (i.e., the universe 1 with the class a removed). Today this class (or set) is referred to as 'the complement of a' and denoted in various ways, including $\overline{a}$, a', and a^c.

(b) Later in his paper [5, p. 194], Huntington used the postulates of the first postulate set to prove that for each $a \in K$, the element $\overline{a}$ in Postulate V not only exists, but is also uniquely determined by a. In other words, given an element $a \in K$, there is only one element in K that 'acts' like the element $\overline{a}$ defined by Postulate V; thereafter, Huntington referred to this element $\overline{a}$ as *the supplement of a*.

To prove that the supplement of a is unique, where $a \in K$ is given, Huntington assumed that $\overline{a}_1, \overline{a}_2 \in K$ were two elements that 'acted' like the supplement of a — that is, he assumed that $a \oplus \overline{a}_1 = \bigvee$ and $a \odot \overline{a}_1 = \bigwedge$ (so that $\overline{a}_1$ 'acts' like the supplement of a) and he assumed that $a \oplus \overline{a}_2 = \bigvee$ and $a \odot \overline{a}_2 = \bigwedge$ (so that $\overline{a}_2$ 'acts' like the supplement of a). He then proved that $\overline{a}_1 = \overline{a}_2$. In other words, $\overline{a}_1$ and $\overline{a}_2$ are, in fact, the same element.

Symbolically, we can represent this argument as follows:

$$(\forall a, \overline{a}_1, \overline{a}_2 \in K) \left[\left(a \oplus \overline{a}_1 = a \oplus \overline{a}_2 = \bigvee \right) \wedge \left(a \odot \overline{a}_1 = a \odot \overline{a}_2 = \bigwedge \right) \rightarrow \overline{a}_1 = \overline{a}_2 \right]$$

Use this statement as a model to write a symbolic version of an argument for proving the additive identity $\bigwedge$ defined by Postulate IIa is unique. Then do the same for the multiplicative identity $\bigvee$ defined by Postulate IIb.

(c) As noted in part (b) of this question, the postulates of Huntington's first postulate set imply that every element of K has a unique supplement in K. Given $b, c \in K$, note that we can use this uniqueness property for supplements to prove that $c = \overline{b}$ simply by showing that c satisfies the following simultaneous system of equations[13]:

$$b \oplus c = \bigvee \quad \text{and} \quad b \odot c = \bigwedge$$

Use this idea and commutativity of $\oplus$, $\otimes$ (Postulates IIIa , b) to *prove* the following claim:

For every $a \in K$, $\overline{\overline{a}} = a$.

That is, show that given any $a \in K$, the supplement of the element $\overline{a}$ is, in fact, a itself. Begin your proof by introducing an arbitrary $a \in K$ and using Postulate V to obtain $\overline{a}$.

(d) The quantifier '∃!' [read 'there exists a unique . . . '] signifies both existence and uniqueness. Use this quantifier and any other necessary logical symbols (see footnote 11, page 6) to write Postulate V in symbolic form. (Note: You will need to incorporate the symbolic forms of Postulates IIa and IIb from project question 3c, page 6, into this statement.)

You will have noticed that Huntington's first postulate set actually contains ten postulates, eight of which are arranged as dual pairs (e.g., Ia , Ib). Examine these dual pairs carefully before responding to the following question.

5. (a) Write a short description of how to obtain the dual of a statement.
Illustrate your method by writing the dual of the following equalities.[14]

(i) $a \odot a = a$ (ii) $a \odot \bigwedge = \bigwedge$ (iii) $a \oplus (a \odot b) = a$

(b) Use your method to write the dual of Postulate V, and comment on why Huntington did not include this dual statement separately.

[13] It is important to show that c satisfies *both* equations before concluding that $c = \overline{b}$. Given any particular class b in the concrete interpretation of the postulates as statements about classes, it is easy to find multiple examples of classes c which satisfy one but not both equations. Only the unique supplement of b (i.e., the class not-b) will satisfy both equations simultaneously. The reader is encouraged to examine specific examples of classes to see this.

[14] We will see these equalities again as theorems later in Huntington's paper.

The fact that both a statement and its dual will be true in the algebra of symbolic logic had also been noted by several earlier mathematicians. Huntington referred to the principle of duality as 'a characteristic feature of the algebra' [5, p. 294]. Based on this principle, once Huntington included the familiar distributive property of multiplication over addition [$a(b+c) = ab + ac$] as a postulate (IVa), it was important for him to also include the dual property of distributivity of addition over multiplication [$a+(bc) = (a+b)(a+c)$] as a postulate (IVb). Despite the fact that this latter property fails in standard algebra, previous mathematicians had already shown that it does hold within the concrete interpretation of operations on classes. Given his interest in providing a completely abstract treatment, however, Huntington could not use a concrete interpretation as justification. Instead, he assigned both distributive properties the status of *postulates*, or *axioms* — statements which are assumed without proof.

In contrast to his treatment of the dual distributive properties, however, Huntington did *not* include the associative properties for $\oplus$ and $\odot$ as basic axioms in his first postulate set, although these too had been recognized by earlier mathematicians as valid in the logic of classes. Rather, Huntington *deduced* associativity and other known properties the logic of classes as formal consequences of his postulates. The following excerpt includes his list of basic theorems which follow from his first postulate set, along with one of his proofs [5, p. 293-295]. Although Huntington omits the (universal) quantifiers in stating these properties, note that these are general properties which hold for all elements $a, b \in K$.

VIIa. The element $\bigwedge$ in IIa is unique: $a \oplus \bigwedge = a$.

VIIb. The element $\bigvee$ in IIb is unique: $a \odot \bigvee = a$.

VIIIa. $a \oplus a = a$

VIIIb. $a \odot a = a$

IXa. $a \oplus \bigvee = \bigvee$

IXb. $a \odot \bigwedge = \bigwedge$

Xa. $a \oplus (a \odot b) = a$ (The "law of absorption.")

Xb. $a \odot (a \oplus b) = a$

XI. The element $\overline{a}$ in V is uniquely determined by a: $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$.

XIIa. $a \oplus b = \overline{\overline{a} \odot \overline{b}}$, and

XIIb. $a \odot b = \overline{\overline{a} \oplus \overline{b}}$.

XIIIa. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. (Associative law for addition.)

XIIIb. $(a \odot b) \odot c = a \odot (b \odot c)$. (Associative law for multiplication.)

... ...

In the following proofs we write, for brevity, $a \odot b = ab$. The proofs for the theorems "b" may be obtained from the proofs for the corresponding theorems "a" by interchanging $\oplus$ with $\odot$ and $\bigwedge$ with $\bigvee$.

... ...

Proof of VIIIa. By V (in view of VIIa, b) take $\overline{a}$ so that $a \oplus \overline{a} = \bigvee$ and $a\overline{a} = \bigwedge$. Then by Ia, IIa, b, and IVa we have

$$a \oplus a = (a \oplus a) \bigvee = (a \oplus a)(a \oplus \overline{a}) = a \oplus (a\overline{a}) = a \oplus \bigwedge = a$$

6. This question examines Huntington's proofs of the dual Idempotent Properties (VIIIa, b) in more detail.

   (a) Note that Huntington began his proof of Property VIIIa (Idempotency for $\oplus$) by introducing the element $\overline{a}$, indicating that he is allowed to do so 'By V (in view of VIIa, b)'. Although not included in the previous excerpt, Huntington did provide proofs for Properties VIIa, b (uniqueness of $\bigvee$, $\bigwedge$)[15], prior to proving property VIIIa.

   After reviewing Postulate V, write a paragraph to explain why it was necessary for Huntington to establish Properties VIIa and VIIb *before* he introduced $\overline{a}$ into his proof of Property VIIIa. (It may also be useful to review question 4b, pp. 6-7.)

   (b) Now consider each step of the equation which shows that $a \oplus a = a$ and indicate which postulate (Ia, IIa, IIb, or IVa) was used at each step. (Huntington listed the necessary postulates in numerical order, not in the order in which they were actually used.)

   (c) Finally, provide a proof for Property VIIIb (Idempotency for Multiplication), indicating the required postulate for each step of your proof. (Feel free to modify the proof format as desired to make the proof more 'readable.')

We continue our reading with Huntington's proof [5, p. 295] of Property Xa, which states that $a \oplus (a \odot b) = a$. This is one of two laws commonly referred to as 'absorption.' The dual absorption law, $a \odot (a \oplus b) = a$, is stated in Huntington's Property Xb. **Recall that Huntington writes ab as an abbreviation for $a \odot b$ in his proofs.**

Proof of Xa. By Ia , b, IIb, IVb, IIIa and IXa we have

$$a \oplus (a \odot b) = (a \bigvee) \oplus (ab) = a(\bigvee \oplus b) = a(b \oplus \bigvee) = a \bigvee = a$$

7. (a) In his proof of Absorption Property Xa, Huntington did list the necessary postulates in essentially the same order in which they were actually used in the equation; verify that this is the case. Then provide a similar proof for Absorption Property Xb.

   (b) Now consider the interpretation of these two properties as statements about classes. Adopting notation for class operations similar to that used by Boole[16], property Xa would be written as "$a + ab = a$". Explain why this property makes sense when interpreted as a statement about classes. Do the same with Property Xb, after first translating it into notation based on that used by Boole.

We close this section with two questions which examine Huntington's proofs of uniqueness (outlined in project question 4b, page 7) in more detail. We begin with his proofs of uniqueness for the additive and multiplicative identities ($\bigwedge$, $\bigvee$), as stated in Properties VIIa, b [5, p. 294]. **Again recall that Huntington writes ab as an abbreviation for $a \odot b$ in his proofs.**

[15] The proofs for Properties VIIa, b will be examined in project question 8, page 10

[16] We note again that Boole himself restricted + to disjoint classes only; our own use of + follows that of Venn and other later mathematicians. See footnote 2, p. 1 of this project

Proof of VIIa. Suppose there were two elements, $\bigwedge_1$ and $\bigwedge_2$, such that $a \oplus \bigwedge_1 = a$ and $a \oplus \bigwedge_2 = a$ for every element a. Then, putting $a = \bigwedge_2$ in the first equation and $a = \bigwedge_1$ in the second, we should have $\bigwedge_2 \oplus \bigwedge_1 = \bigwedge_2$ and $\bigwedge_1 \oplus \bigwedge_2 = \bigwedge_1$; whence, by IIIa, $\bigwedge_1 = \bigwedge_2$.

8. (a) Write a paragraph summarizing Huntington's approach to formally proving the uniqueness of the additive identity $\bigwedge$ (Property VIIa). Identify clearly all postulates (by name or by number) and other mathematical properties that he needed to carry out this approach.
   (b) Modify Huntington's proof of uniqueness of the additive identity $\bigwedge$ (Property VIIa) to prove uniqueness for the multiplicative identity $\bigvee$ (Property VIIb).

We now look at Huntington's proof of uniqueness for supplements, as stated in Properties XI [5, p. 295]. **Again recall that Huntington writes ab as an abbreviation for $a \odot b$ in his proofs.**

Proof of XI. Suppose that for a given element a there were two elements, $\overline{a}_1$ and $\overline{a}_2$, such that $a \oplus \overline{a}_1 = a \oplus \overline{a}_2 = \bigvee$ and $a \odot \overline{a}_1 = a \odot \overline{a}_2 = \bigwedge$; then using Ia,b, IIa, b, IIIb, IVb, and V we should have

$$\overline{a}_2 = \bigvee \overline{a}_2 = (a \oplus \overline{a}_1)\overline{a}_2 = (a\,\overline{a}_2) \oplus (\overline{a}_1\,\overline{a}_2) = \bigwedge \oplus (\overline{a}_1\;\overline{a}_2)$$

$$= (\overline{a}_1\,a) \oplus (\overline{a}_1\,\overline{a}_2) = \overline{a}_1(a \oplus \overline{a}_2) = \overline{a}_1 \bigvee = \overline{a}_1$$

9. (a) Provide a reason for each step of the equation which shows that $\overline{a}_1 = \overline{a}_2$ in this proof.
   (b) Write a paragraph commenting on the differences and similarities in approach between Huntington's proof of uniqueness for the additive identity $\bigwedge$ (Property VIIa) and his proof of uniqueness for the supplement $\overline{a}$ (Property XI).

3 Properties of Axiom Systems: Consistency and Independence

Huntington's purpose in providing formal proofs of theorems VIIa – XIIIb was not merely to establish that they followed from the given postulates. Rather, he was interested in using these theorems to establish the equivalence of the three different postulate sets which he studied in the paper. Both the second and third set of postulates given by Huntington began with only one operation or relation [$\lessdot$ and $\oplus$ respectively], along with elements $\bigvee$, $\bigwedge$ and $\overline{a}$ defined by the postulates. The remaining operation(s) and/or relation were then defined in terms of these. This was also how Huntington treated the dyadic relation $\lessdot$ in his first postulate set, defining that relation in terms of $\oplus$, $\odot$ and supplements [5, p. 294].[17] In his third set of postulates, where $\oplus$ served as the given (undefined) operation, Huntington defined 'times' in terms of 'plus' as follows:

$$a \odot b = \overline{\overline{a} \oplus \overline{b}}.$$

Dually, if one began with postulates based only on $\odot$, one could then define $\oplus$ by the dual equality, $a \oplus b = \overline{\overline{a} \odot \overline{b}}$. In either case, one could then deduce the properties of the *defined* operation from its definition and the properties stipulated for the given (undefined) operation by the postulate set.

Recall that the equations $a \odot b = \overline{\overline{a} \oplus \overline{b}}$ and $a \oplus b = \overline{\overline{a} \odot \overline{b}}$ (known today as DeMorgan's Laws) put forward as *definitions* within the alternative postulate sets discussed in the previous paragraph were instead established as *theorems* (XIIa,b) within Huntington's first postulate set. On the other hand, certain other equalities which served as *axioms* in his first postulate set, such as commutativity of addition, were not explicitly stipulated as axioms in his second or third postulate set, and therefore required proofs in order to be established as *theorems* within those systems. Ultimately, Huntington established that the complete set of algebraic properties in its totality is the same regardless of the postulate set used (therefore ensuring the same underlying structure in each case), but the status of these properties (definition versus axiom versus theorem) was different in each.

This freedom to choose among alternative postulate sets for the same structure naturally leads to the question of how to select a particular postulate set with which to work from among the alternatives. We consider this issue in project questions 10 and 11 below.

10. What advantages and disadvantages might be obtained in constructing an axiom system by the use of a smaller number of fundamental concepts (e.g., one operation instead of two)? What about the number of postulates: is it desirable to limit the number of postulates used in an axiom system? Why or why not?

The question of what constitutes a 'good' postulate system is also one that Huntington considered in this paper, citing three desirable characteristics, but concluding essentially that there is no criterion for selecting a 'best' postulate system [5, p. 290]:

> These postulates are simply *conditions* arbitrarily imposed on the fundamental concepts and must not, of course, be inconsistent among themselves. Any set of *consistent* postulates would give rise to a corresponding algebra, — namely the totality of propositions which follow from these postulates by logical deduction. For the sake of elegance, every set of postulates should be free from redundancies; in other words, the postulates of every set should be *independent*, no one of them deducible from the rest. For, if any one of the postulates were a consequence of the others, it should be counted among the derived, not among the fundamental propositions. Furthermore, each postulate should be as nearly as possible a *simple statement*, not decomposable into two or more parts; but the idea of a simple statement is a very elusive one, which has not yet been satisfactorily defined, much less attained.
>
> In selecting a set of consistent, independent postulates for any particular algebra, one has usually a considerable freedom of choice; several different sets of independent postulates (on a given set of fundamental concept) may serve as the basis of the same algebra the only logical requirement is that every such set of postulates must be deducible from every other.

[17] This definition has not been included in this project.

11. Re-examine Huntington's first postulate set (page 5) and comment on whether the axioms included there constitute 'simple statements.' What other characteristics, if any, do you think a 'good' postulate system should possess besides those mentioned by Huntington?

Huntington himself said nothing further in this paper about the criteria of 'simplicity' for postulates. He did, however, prove that each of his three postulate sets is both consistent and independent. The basic idea behind the technique he used to do so is to introduce a concrete system of interpretation, and check that all the postulates of the system necessarily hold within that interpretation. As Huntington described in the first paragraph below [5, p. 293], all properties of the abstract system (both postulates and theorems) will necessarily hold in that system of interpretation, allowing us to conclude that the postulate set itself is consistent. In the remainder of this excerpt, Huntington described two particular concrete systems of interpretation for his first postulate set, thereby establishing its consistency.

> To show the consistency of the postulates, we have only to exhibit some system $(K, \oplus, \odot)$ in which K, $\oplus$, and $\odot$ are so interpreted that all the postulates are satisfied. For then the postulates themselves, and all their consequences, will be simply expressions of the properties of this system, and therefore can not involve contradiction (since no system which really exists can have contradictory properties).
>
> One such system is the following: $K=$ the class of regions in the plane including the "null region" and the whole plane; $a \oplus b =$ the "logical sum" of a and b (that is, the smallest region which includes them both); $a \odot b =$ the "logical product" of a and b (that is, the largest region which lies within them both).
>
> Another such system, in fact the simplest possible one, is this: $K =$ the class comprising only two elements, say 0 and 1, with $\oplus$ and $\odot$ defined by the tables

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

Today, systems of interpretations such as those put forward in the preceding excerpt are referred to as 'models' of an axiom system. In Huntington's time, the use of models to prove consistency and independence was a relatively new method; today, the use of models plays a central role in set theory and mathematical logic. In essence, the first model described by Huntington is the Venn diagram representation of Boole's logic of classes, with $\oplus$ and $\odot$ representing the set operations of union and intersection respectively. The second model given by Huntington is the special arithmetical algebra on 0 and 1 which was initially suggested to Boole by the Idempotent Law ($x^2 = x$) in his study of the logic of classes. Note that, as a statement about classes, it makes sense to assert $1 \oplus 1 = 1$ in the given table since the aggregate (or union) of any set with itself will be the set itself. Huntington, however, presented this special two-valued algebra as a purely abstract system, with no reference to classes or class operations. Thus, in showing that this system is a model of the first postulate set, there can be no reference to classes or class operations. Although Huntington omitted this proof from his paper, we consider it here in detail.

To establish that $K = \{0\,,\,1\}$ under the proposed operations is a model of Huntington's first postulate system, one must verify that this system satisfies all ten of the given postulates. It is clear from the operation tables that the values of $a \oplus b$ and $a \odot b$ (either 0 or 1) are in K for any given values of a and b from K; thus, K is *closed* under $a \oplus b$ and $a \odot b$ and postulates Ia and Ib are satisfied. Similarly, postulate VI is clearly satisfied since 0 and 1 are distinct elements of K. It is also straightforward (albeit tedious) to directly verify both versions of Postulates III and IV (commutativity and distributivity) by examining all possible cases for values of a, b and c in these postulates.[18] This leaves only Postulates II and V to verify; the next two project questions consider these two properties.

12. Prove that the model defined by $K = \{0\,,\,1\}$ and the operation tables given on page 12 satisfies Postulate II (existence of identities) by showing that setting $\bigwedge = 0$ and $\bigvee = 1$ satisfies the conditions given in Postulates IIa and IIb respectively.

    Then explain why we are sure, based solely on the given operation tables rather than by invocation of theorem VII, that the elements $\bigvee$ and $\bigwedge$ are unique. That is, use the operation tables to show that 0 is not a multiplicative identity and 1 is not an additive identity for this system. Why does this suffice to establish uniqueness of the identity elements in this system?

13. In project question 12, it was shown that unique identities for $\oplus$ and $\odot$ exist in the model defined by $K = \{0\,,\,1\}$ and the given tables, so that the hypothesis of Postulate V holds in this model. Prove this model satisfies Postulate V (existence of supplements, given unique identity elements) by showing that there are elements $\overline{0}$ and $\overline{1}$ in K with the properties required by the conclusion of that postulate. (Use $\bigwedge = 0$ and $\bigvee = 1$, as was established project question 12.)

Huntington's proof of independence also made use of the notion of a model, as described in the following excerpt [5, p. 296].

∞∞∞∞∞∞∞∞

The ten postulates of the first set are *independent*; that is, no one of them can be deduced from the other nine. To show this, we exhibit, in the case of each postulate, a system $(K, \oplus, \odot)$ which satisfies all the other postulates, but not the one in question. This postulate, then, cannot be a consequence of the others; for it it were, every systems which had the other properties would have this property also, which is not the case.

For postulate VI take $K =$ the class comprising a single element, a, with $a \oplus a = a$ and $a \odot a = a$.

∞∞∞∞∞∞∞∞

Notice how the model for the independence of Postulate VI (existence of at least two elements) given in the second paragraph of this excerpt illustrates the general process for proving independence described by Huntington in the first paragraph. The underlying class for that model consists of the singleton set $K = \{a\}$, where closure under both operations is guaranteed by defining these operations as $a \oplus a = a$ and $a \odot a = a$. Commutativity and associativity for both operations follows from the

[18] See Appendix A for one approach to verifying Postulates III and IV (commutativity and distributivity). Detailed formal proofs for Postulates I and VI are also included in the appendix.

fact that the result of any sequence of operations will be a. It is equally clear that both operations possess an identity element ($\bigwedge = \bigvee = a$), and that the only element in K is its own supplement ($\overline{a} = a$). Thus, the only axiom which fails is the one which stipulates that K must contain *two* distinct elements.

Although finding and verifying a model for a particular postulate system is often more complicated than this, the basic idea of any independence proof is the same: plan for limited failure by assigning particular meanings to the fundamental concepts (e.g, K, $\oplus$, $\odot$) in such a way that exactly one of the given postulate fails to hold. The table given by Huntington in continuation of this excerpt below [5, p. 296] illustrates how this can be done for each of the first nine postulates of his first set. The project questions which follow this excerpt examine these models in more detail.

∞∞∞∞∞∞∞∞

For the other postulates, take K = a class containing two elements, say 0 and 1, with $\oplus$ and $\odot$ defined appropriately for each case, as indicated in the following scheme:

	$0\oplus 0$	$0\oplus 1$	$1\oplus 0$	$1\oplus 1$	$0\odot 0$	$0\odot 1$	$1\odot 0$	$1\odot 1$
Ia	0	1	1	x	0	0	0	1
Ib	0	1	1	1	x	0	0	1
IIa	0	0	0	0	0	0	0	1
IIb	0	1	1	1	1	1	1	1
$IIIa$	0	0	1	1	0	0	0	1
$IIIb$	0	1	1	1	0	0	1	1
IVa	0	1	1	0	0	0	0	1
IVb	0	1	1	1	1	0	0	1
V	0	1	1	1	0	1	1	1

In verifying these results, notice that the system for IIa (or IIb) satisfies postulate V "vacuously," since no element having the properties of $\bigwedge$ (or $\bigvee$) exists; while the system for IIIa (or IIIb) also satisfies V vacuously, since the element $\bigwedge$ (or $\bigvee$) is not uniquely determined. In other systems, $\bigwedge = 0$ and $\bigvee = 1$, expect in the system for V, where $\bigwedge = 0$ and $\bigvee = 0$.

∞∞∞∞∞∞∞∞

Note that each line of this table defines both $\oplus$ and $\odot$ for a particular model. For example, in the system labeled Ia, these two operations are defined as follows:

$\oplus$	0	1
0	0	1
1	1	x

$\odot$	0	1
0	0	0
1	0	1

Similarly, the operation tables for the system labeled IIIb could be written as follows:

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	1	1

14. (a) Explain how we know that postulate Ia (closure under $\oplus$) fails in the system labeled Ia, where $K = \{0, 1\}$. (You do not need to check that the other axioms hold.)

    (b) Explain how we know that postulate IIIa (commutativity of $\oplus$) fails in the system labeled IIIa. (You do not need to check that the other axioms hold.)

    (c) In his model of the first postulate set (page 12), Huntington set $1 \oplus 1 = 1$. In the system labeled IVa above, he assigned the value '0' to the expression '$1 \oplus 1$'. Verify that Postulate IVa (distributivity of multiplication over addition) fails under this assignment by comparing the values of the expressions '$a \oplus (b \odot c)$' and '$(a \oplus b) \odot (a \oplus c)$' for the specific values of $a = 1$, $b = 1$ and $c = 0$. (You do not need to check the other axioms.)

    (d) Find an example to prove that Postulate IVb (distributivity of addition over multiplication) fails in the system labeled IVb. (You do not need to check the other axioms.)

15. This question examines Huntington's remarks concerning Postulate V in the final paragraph of the excerpt on page 14. Re-read that paragraph, and recall that Postulate V, stated below for convenience, has the form of a conditional statement:

    V. If the elements $\bigwedge$ and $\bigvee$ in postulates IIa and IIb exist and are unique, then for every element a there is an element $\overline{a}$ such that $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$.

    (a) Consider the system labeled IIa.

    Verify Huntington's claim that this system contains '...no element having the properties of $\bigwedge$ ...' by showing that neither 0 nor 1 acts as an additive identity. That is, let $\bigwedge = 0$ and show that $a \oplus \bigwedge = a$ fails for some value of a. Then do the same with $\bigwedge = 1$.

    Recall that Postulate IIa asserts the existence of an additive identity $\bigwedge$. This lack of an additive identity proves that Postulate IIa fails in this system. Why can we now also conclude that this system ' ...satisfies postulate V "vacuously" ...?

    (b) Now consider the system labeled IIIa.

    Verify Huntington's claim that '...the element $\bigwedge$ ...is not uniquely determined ...' in this system by showing that BOTH 0 nor 1 act as an additive identity. That is, let $\bigwedge = 0$ and show that for every $a \in K$, we have $a \oplus \bigwedge = a$. Then do the same with $\bigwedge = 1$.

    Why can we now conclude that this system ' ...satisfies postulate V "vacuously" ...?

    (c) Finally, consider the system labeled V.

    Verify Huntington's claim that '...$\bigwedge = 0$ and $\bigvee = 0$ ......' in this system by showing that 0 acts as both an additive identity and a multiplicative identity in this system.

    Explain why this is *not* enough to conclude that $\bigwedge$ and $\bigvee$ are non-unique.
    (We will return to this idea in project question 16 below.)

16. This question explores the status of Postulate V (stated above) in the system labeled V further.

    (a) Note that the hypothesis of Postulate V requires both the existence and the uniqueness of $\bigwedge$ and $\bigvee$. In project question 15c, existence was established by showing that we can let $\bigwedge = 0$ and $\bigvee = 0$. Now establish uniqueness of $\bigwedge$ by showing that 1 does *not* act as an additive identity in this system. Then establish uniqueness of $\bigvee$ in the same way.

    (b) Since the hypothesis of Postulate V holds in this system by part (a), explain how we know that postulate V fails to hold in this system. In doing so, remember that $\bigwedge = 0$ and $\bigvee = 0$ in this system (as shown in project question 15c).

4 Epilogue: Boolean Algebra Today

Today, the study of boolean algebra remains an important subject, both from a theoretical point of view for mathematicians and from a practical point of view for computer scientists and engineers. For both sorts of practitioners, it is standard to start with a set of axioms which gives, in a manner of speaking, the 'rules of the game.' A standard axiomatization in an undergraduate textbook today, in fact, bears a strong resemblance to the one studied in this project. We offer one such definition on the following page. As you read this definition, compare it to that given by Huntington in his first postulate set.

> *Definition* A *boolean algebra* is a six-tuple $(S\,, +\,, \cdot\,, '\,, 0\,, 1)$, where S is a set, $+$, $\cdot$ are binary operations on S, $'$ is a unary operation on S and 0, 1 are elements of S, subject to the following eight laws:
>
> 1. Commutativity of Addition: $(\forall x, y \in S)(x + y = y + x)$
> 2. Commutativity of Muliplication: $(\forall x, y \in S)(x \cdot y = y \cdot x)$
> 3. Associativity of Addition: $(\forall x, y, z \in S)((x + y) + z = x + (y + z))$
> 4. Associativity of Multiplication: $(\forall x, y, z \in S)((x \cdot y) \cdot z = x \cdot (y \cdot z))$
> 5. Distributivity of Multiplication over Addition: $(\forall x, y, z \in S)(x \cdot (y + z) = x \cdot y + x \cdot z)$
> 6. Distributivity of Addition over Multiplication: $(\forall x, y, z \in S)(x + y \cdot z = (x + y) \cdot (x + z))$
> 7. Identity Laws: $(\forall x \in S)(x + 0 = x \,\wedge\, x \cdot 1 = x)$
> 8. Complement Laws: $(\forall x \in S)(x + x' = 1 \,\wedge\, x \cdot x' = 0)$

One notable difference between Huntington's postulates and today's definition relates to the notation of closure. As indicated in Huntington's footnote 8 (page 4 of this project), the definition of 'rule of combination $\circ$ on a class K' in usage at that time did *not* require that $a \circ b$ to be an element K; thus, Huntington needed additional postulates to ensure closure of K under his two operations. Today, however, the definition of a 'binary operation $\circ$ on a set S' *does* require S to be closed under $\circ$. In one sense, Huntington's closure postulates are still present — they simply make an earlier appearance by being cast in a different light. Today's definition of 'unary operation' (applied to a single element of the set) similarly requires closure, a fact which accounts in part for the difference between Huntington's Postulate V and the Complement Laws of today's definition. Of course, the *relationships* between the complement (Huntington's supplement) and the two identities stated in the Complement Laws still *does* need to be stipulated. However, this no longer needs to be done as part of a complicated conditional statement.

The inclusion of the Associativity Laws marks the other notable difference between Huntington's postulates and today's standard definition of a boolean algebra. It is, of course, still possible to prove associativity from the other axioms — the fundamentals of the mathematics has not changed since Huntington's time! However, the proof is not completely straight forward, so that associativity is often just assumed as an axiom.[19] Note that this means the axiom set is no longer independent, so that any model which satisfies laws 1, 2 and 5–8 will necessarily satisfy both 3 and 4 as well. Including these two extra axioms also means there are two more conditions to verify in establishing that one has a model of the boolean algebra axioms.

With the exception of these differences, and the superficial differences in notation and terminology, today's formal approach to boolean algebras is much like that developed by Huntington in the early twentieth century. This project closes with several questions based on today's definition as a means of illustrating this observation, and as a means to further solidify an understanding of this interesting and important structure.

[19]**Bonus Project Question:** Prove associativity from the remaining axioms.

NOTE: As it was for Huntington, it is typical today to write ab for $a \cdot b$.

17. State the dual of the following expressions and equalities.
    (a) $(xz + yz')\,z$
    (b) $(x' + y')' = xy$
    (c) $(x + y)\,(x + 1) = x + xy + y$
    (d) If $x + y = x + z$ and $x' + y = x' + z$, then $y = z$.

18. Let $(S\,,+\,,\cdot\,,'\,,0\,,1)$ be a boolean algebra. Write a proof of each of the following properties, based on the format modelled in the example given below.

    Bound Law[20] for Addition: For all $x \in S$, $x + 1 = 1$.

    Proof Let $(S\,,+\,,\cdot\,,'\,,0\,,1)$ be a Boolean Algebra and let $x \in S$. We show that $x + 1 = 1$ as follows:

$$\begin{aligned} x + 1 &= (x+1)\cdot 1 && \text{by the Identity Property for Multiplication} \\ &= (x+1)\cdot(x+x') && \text{by the Complement Property for Addition} \\ &= x + 1\cdot x' && \text{by the Distributivity of Addition over Multiplication} \\ &= x + x'\cdot 1 && \text{by Commutativity of Multiplication} \\ &= x + x' && \text{by the Identity Property for Multiplication} \\ &= 1 && \text{by the Complement Property for Addition} \end{aligned}$$

    (a) Bound Law for Multiplication: For all $x \in S$, $x \cdot 0 = 0$
    (b) Uniqueness of Complements: For all $x, y \in S$, if $x + y = 1$ and $xy = 0$, then $y = x'$. Note: This can be also done by adapting Huntington's proof of uniqueness of $\overline{a}$ (page 10).
    (c) Involution Law: For all $x \in S$, $(x')' = x$ *(See Project Question 4c.)*
    (d) Absorption Law for Addition: For all $x, y \in S$, $x + xy = x$ *(See Project Question 7.)*
    (e) Absorption Law for Multiplication: For all $x, y \in S$, $x(x + y) = x$
    (f) Idempotent Law for Addition: For all $x \in S$, $x + x = x$
    (g) Idempotent Law for Multiplication: For all $x \in S$, $x \cdot x = x$
    (h) DeMorgan's Law for Addition: For all $x, y \in S$, $(x + y)' = x'y'$
    (Hint? Show that $(x + y)$ 'acts' like the complement of xy, and invoke uniqueness.)
    (i) For all $x, y \in S$ $(x + y)\,(x + 1) = x + xy + y$.
    (j) For all $x, y \in S$, if $x + y = x + z$ and $x' + y = x' + z$, then $y = z$.
    (k) For all $x, y \in S$, if $x + y = x + z$ and $xy = xz$, then $y = z$.

[20]Both bound laws were identified by Huntington as theorems (IXa,b) of his first postulate set.

19. (a) Let $S = \{1, 2, 3, 6\}$ with the operations $+$, $\cdot$ and $'$ defined on S as follows:

$$\text{Given } x, y \in S, \text{let } x + y = \text{lcm}(x, y) , \; x \cdot y = \text{gcd}(x, y) \text{ and } \; x' = \frac{6}{x}.$$

Show that $(S, +, \cdot, ', 1, 6)$ is an boolean algebra.
That is, verify that $(S, +, \cdot, ', 1, 6)$ satisfies axioms 1 - 8 on page 16 of this project.

NOTE 1: The order of the elements in the six-tuple is important! In particular, this order suggests the additive identity will be 1, and the multiplicative identity will be 6.

NOTE 2: You may find it useful to complete the following tables.

$+$	1	2	3	6
1				
2				
3				
6				

$\cdot$	1	2	3	6
1				
2				
3				
6				

x	1	2	3	6
x'				

(b) Let $S = \{1, 2, 4, 8\}$ with the operations $+$, $\cdot$ and $'$ defined on S as follows:

$$\text{Given } x, y \in S, \text{let } x + y = \text{lcm}(x, y) , \; x \cdot y = \text{gcd}(x, y) \text{ and } \; x' = \frac{8}{x}.$$

Show that $(S, +, \cdot, ', 1, 8)$ is NOT a boolean algebra.

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5 Notes to the Instructor

This project is designed for an introductory or intermediate course in discrete or finite mathematics that considers boolean algebra from either a mathematical or computer science perspective. Some or all of the project could also be used in a course on abstract algebra or model theory. The project does assume some (minimal) familiarity with the set operations of union and intersection and their representation by Venn diagrams. This pre-requisite material may be gained by completing the companion (Boole) project described below, through reading a standard textbook treatment of elementary set operations, or via a short class discussion/lecture.

Although primarily based on an early paper in the study of axiomatizations of boolean algebras, the introductory section of this project provides a concise overview of George Boole's original work on 'the logic of classes' in order to provide students with a connection to a concrete example of a boolean algebra on which they can draw. In section 2, the goal of formal axiomatics is introduced through select readings from Huntington's 1904 paper *Sets of Independent Postulates for the Algebra of Logic*. Section 2 also introduces the now standard axioms for the boolean algebra structure and illustrates how to use these postulates to prove boolean algebra basic properties. Specific project questions also provide students with practice in using symbolic notation, and encourage them to analyze the logical structure of quantified statements. In Section 3, the use of models is illustrated in Huntington's use of the two-valued Boolean algebra on $K = \{0, 1\}$ — first studied by George Boole in his work on the logic of classes — to establish the *independence* and *consistency* of one of his postulate sets. The final section of this project discusses modern (undergradute) notation and axioms for boolean algebras, and provides several practice exercises to reinforce the ideas developed in the earlier sections.

Two other projects on boolean algebra are available as companions to this project, either or both of which could also be used independently of this project. The first companion project "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce," is suitable as a preliminary to the Huntington project. Without explicitly introducing modern notation for operations on sets (until the concluding section), that project develops a modern understanding of these operations and their basic properties within the context of early efforts to develop a symbolic algebra for logic. By steadily increasing the level of abstraction, that project also lays the ground work for a more abstract discussion of a boolean algebra as a discrete structure. Other project questions prompt students to explore a variety of other mathematical themes, including the notion of an inverse operation, issues related to mathematical notation, and standards of rigor and proof.

The second companion project "Applying Boolean Algebra to Circuit Design: Claude Shannon" is suitable as either as a preliminary to or as a follow-up to the Huntington project on axiomatization. Based on Shannon's groundbreaking paper *A Symbolic Analysis of Relay and Switching Circuits*, that project begins with a concise overview of the two major historical antecedents to Shannon's work: Boole's original work in logic and Huntington's work on axiomatization. The project then develops standard properties of a boolean algebra within the concrete context of circuits, and provides students with practice in using these to simplify boolean expressions The two-valued boolean algebra on $K = \{0, 1\}$ again plays a central role in this work. The project closes with an exploration the concept of a 'disjunctive normal form' for boolean expressions, again within the context of circuits.

Implementation with students of any of these projects may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. For the Huntington project, the instructor is encouraged to provide students with copies of Huntington's axioms and the definitions of his various models for consistency and independence on a separate sheet; these are included in Appendices B and C of this project.

Appendix A: Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization
Proving Consistency of First Set of Postulates - Huntington

CLAIM: Let $K = \{0, 1\}$ and define the operations $\oplus$ and $\odot$ on K as indicated in the following tables:

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

Then $(K, \oplus, \odot)$ is a model of Huntington's postulates Ia,b , IIa,b , IIIa,b , IVa,b , V and VI.

PROOF: Let $K = \{0, 1\}$ and define the operations $\oplus$ and $\odot$ on K as indicated in the tables above.
We verify that $(K, \oplus, \odot)$ satisfies Huntington's postulates Ia,b , IIIa,b ,IVa,b ,and VI below.
Postulates IIa,b are verified in project question 12 (page 13), and Postulate V in project question 13 (page 13).

- Postulates Ia,b: *For all elements* $a, b \in K$, $a \oplus b$ *and* $a \odot b$ *are elements of* K.

  Given $a, b \in K$, a and b can only take on the values 0 and 1. Examining the given tables, we see that in every possible case, the values of $a \oplus b$ and $a \odot b$ are also either 0 or 1. This proves that $(K, \oplus, \odot)$ satisfies Postulates Ia and Ib.

- Postulates IIIa,b: *For all* $a, b \in K$, *if* $a \oplus b$, $b \oplus a$, $a \odot b$, $b \odot a \in K$, *then* $a \oplus b = b \oplus a$ *and* $a \odot b = b \odot a$.

  By Postulates Ia & b, we know $a \oplus b$, $b \oplus a$, $a \odot b$ and $b \odot a$ are in K whenever a, b are in K. Thus, we must verify that the desired equalities holds for every possible assignment of the values of 0 and 1 to a , b. The following table demonstrates this is true, thereby proving that $(K, \oplus, \odot)$ satisfies Postulates IIIa and IIIb.

a	b	$a \oplus b$	$b \oplus a$	$a \odot b$	$b \odot a$
0	0	0	0	0	0
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	1	1	1

- Postulates IVa,b: *For all* $a, b, c \in K$, *if* $a \oplus b$, $a \oplus c$, $b \odot c$, $a \oplus (b \odot c)$, $(a \oplus b) \odot (a \oplus c)$, $a \odot b$, $a \odot c$, $b \oplus c$, $a \odot (b \oplus c)$, $(a \odot b) \oplus (a \odot c) \in K$, *then* $a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ *and* $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

  By Postulates Ia & Ib, we know $a \oplus b$, $a \oplus c$, $b \odot c$, $a \oplus (b \odot c)$, $(a \oplus b) \odot (a \oplus c)$, $a \odot b$, $a \odot c$, $b \oplus c$, $a \odot (b \oplus c)$, $(a \odot b) \oplus (a \odot c)$ are in K whenever a, b are in K. Thus, we must verify that the desired equalities holds for every possible assignment of the values of 0 and 1 to a , b , c. The following two tables demonstrate this is true, thereby proving that $(K, \oplus, \odot)$ satisfies Postulates IVa and IVb.

$a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$

a	b	c	$b \odot c$	$a \oplus (b \odot c)$	$(a \oplus b)$	$(a \oplus c)$	$(a \oplus b) \odot (a \oplus c)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

a	b	c	$b \oplus c$	$a \odot (b \oplus c)$	$(a \odot b)$	$(a \odot c)$	$(a \odot b) \oplus (a \odot c)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

- Postulates VI: *There are at least two elements,* x *and* y*, in the class such that* $x \neq y$.

  Taking $x = 0$ and $y = 1$, we see that $x \in K$, $y \in K$ and $x \neq y$. This proves that $(K, \oplus, \odot)$ satisfies Postulates VI.

Appendix B: Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization

HUNTINGTON'S FIRST SET OF POSTULATES AND RESULTING THEOREMS

§1. The First Set of Postulates.

Ia. *$a \oplus b$ is in the class whenever a and b are in the class.*

Ib. *$a \odot b$ is in the class whenever a and b are in the class.*

IIa. *There is an element $\bigwedge$ such that $a \oplus \bigwedge = a$ for every element a.*

IIb. *There is an element $\bigvee$ such that $a \odot \bigvee = a$ for every element a.*

IIIa. *$a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ are in the class.*

IIIb. *$a \odot b = b \odot a$ whenever a, b, $a \odot b$, and $b \odot a$ are in the class.*

IVa. *$a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ whenever a, b, $a \oplus b$, $a \oplus c$, $b \odot c$, $a \oplus (b \odot c)$ and $(a \oplus b) \odot (a \oplus c)$ are in the class.*

IVb. *$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ whenever a, b, $a \odot b$, $a \odot c$, $b \oplus c$, $a \odot (b \oplus c)$ and $(a \odot b) \oplus (a \odot c)$ are in the class.*

V. If the elements $\bigwedge$ and $\bigvee$ in postulates IIa and IIb exist and are unique, then for every element a there is an element $\overline{a}$ such that $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$

VI. There are at least two elements, x and y, in the class such that $x \neq y$.

VIIa. The element $\bigwedge$ in IIa is unique: $a \oplus \bigwedge = a$.

VIIb. The element $\bigvee$ in IIa is unique: $a \odot \bigvee = a$.

VIIIa. $a \oplus a = a$

VIIIb. $a \odot a = a$

IXa. $a \oplus \bigvee = \bigvee$

IXb. $a \odot \bigwedge = \bigwedge$

Xa. $a \oplus (a \odot b) = a$ (The "law of absorption.")

Xb. $a \odot (a \oplus b) = a$

XI. The element $\overline{a}$ in V is uniquely determined by a: $a \oplus \overline{a} = \bigvee$ and $a \odot \overline{a} = \bigwedge$.

XIIa. $a \oplus b = \overline{\overline{a} \odot \overline{b}}$, and

XIIb. $a \odot b = \overline{\overline{a} \oplus \overline{b}}$.

XIIIa. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. (Associative law for addition.)

XIIIb. $(a \odot b) \odot c = a \odot (b \odot c)$. (Associative law for multiplication.)

... ...

Proof of VIIIa. By V (in view of VIIa, b) take $\overline{a}$ so that $a \oplus \overline{a} = \bigvee$ and $a\overline{a} = \bigwedge$. Then by Ia, IIa, b, and IVa we have

$$a \oplus a = (a \oplus a)\bigvee = (a \oplus a)(a \oplus \overline{a}) = a \oplus (a\overline{a}) = a \oplus \bigwedge = a$$

HUNTINGTON'S MODEL FOR CONSISTENCY OF THE FIRST SET OF POSTULATES

$K =$ the class comprising only two elements, say 0 and 1, with $\oplus$ and $\odot$ defined by the tables

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

HUNTINGTON'S MODELS FOR INDEPENDENCE OF THE FIRST SET OF POSTULATES

For postulate VI take $K =$ the class comprising a single element, a, with $a \oplus a = a$ and $a \odot a = a$.

For the other postulates, take $K =$ a class containing two elements, say 0 and 1, with $\oplus$ and $\odot$ defined appropriately for each case, as indicated in the following scheme:

	$0 \oplus 0$	$0 \oplus 1$	$1 \oplus 0$	$1 \oplus 1$	$0 \odot 0$	$0 \odot 1$	$1 \odot 0$	$1 \odot 1$
Ia	0	1	1	x	0	0	0	1
Ib	0	1	1	1	x	0	0	1
IIa	0	0	0	0	0	0	0	1
IIb	0	1	1	1	1	1	1	1
$IIIa$	0	0	1	1	0	0	0	1
$IIIb$	0	1	1	1	0	0	1	1
IVa	0	1	1	0	0	0	0	1
IVb	0	1	1	1	1	0	0	1
V	0	1	1	1	0	1	1	1

In verifying these results, notice that the system for IIa (or IIb) satisfies postulate V "vacuously," since no element having the properties of $\bigwedge$ (or $\bigvee$) exists; while the system for IIIa (or IIIb) also satisfies V vacuously, since the element $\bigwedge$ (or $\bigvee$) is not uniquely determined. In other systems, $\bigwedge = 0$ and $\bigvee = 1$, expect in the system for V, where $\bigwedge = 0$ and $\bigvee = 0$.